

Exam III , MTH 221, Summer 2018

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Score = $\frac{34}{37}$

QUESTION 1. (4 points) Let A be a matrix 3×4 ($n \times m$). Given Rank(A) = 3. Convince me that for each point $Q \in R^3$, say $Q = (a, b, c)$, the system of linear equations, $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ has infinitely many solutions.

lives in $\mathbb{R}^3$

A, 3rows, 4 variables

→ Rank(A) = 3 = dim(col(A)) = 3 → It has 3 leaders and 1 free variable

Because there's a free variable → it's of infinitely many solutions

It's consistent because $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ can be written as linear combination of the columns of A of scalars from $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

dim(row(A)) = 3 → Also, it is consistent because all rows of A are certainly nonzero rows so we'll not have the situation 0 = c, therefore, its consistent

since $m > n$ & consistent → ∞ many solutions

-3

QUESTION 2. (4 points) Let A be a 3×2 matrix such that $A \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} - A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \\ 0 & 0 \end{bmatrix}$. Find the matrix A.

$$A\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} - A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \\ 0 & 0 \end{bmatrix}$$

$$A\left(\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & 3 \\ 0 & 0 \end{bmatrix}$$

$$\left[A\left(\underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}}_{Q}\right)\right]Q^{-1} = \left(\underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 3 \\ 0 & 0 \end{bmatrix}}_{3\times 2}\right) \cdot Q^{-1}$$

$$Q^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$$

$$\text{col}_1 A = \frac{1}{2}\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{col}_2 A = 0\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 3 \\ 0 & 0 \end{bmatrix}$$

QUESTION 3. (6 points)

(i) $D = \{A \in R^{2\times 2} \mid |A| = 0\}$. Convince me that D is not a subspace of $R^{2\times 2}$.

$x = \begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} \in D$ $\quad x+y = \begin{bmatrix} 2 & 2 \\ 4 & 3 \end{bmatrix}$ $\quad \det(x+y) = -2$

$y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \in D$ $\quad \therefore x+y \notin D$ → axiom 1 (under addition) fails

(ii) $D = \{f(x) \in P_3 \mid f(-1) = 1\}$. Convince me that D is not a subspace of P_3.

D is not a subspace as 0 (additive identity) does not belong in D

+ axiom 3 fails

(iii) $D = \{(a_1, a_2^2, 0) \mid a_1, a_2 \in R\}$. Convince me that D is not a subspace of R^3. 1

column 2 fails

$x = (1,4,0) \in D$

take α as -1

$\alpha x = (-1, -4, 0) \rightarrow$ not present in D (D only has positive values in 2nd coordinate)

* under scalar multiplication fails

QUESTION 4. (6 points) Let $F = \begin{bmatrix} a & b & 2 & 0 \\ 0 & 0 & c & 3 \\ h & 4 & k & 0 \end{bmatrix}$. Given that F is row-equivalent to $W = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Find the values of a, b, c, d, h, k.

row(F) = span$\{(1,1,1,0), (0,0,1,1)\}$ 　 row(W) = span$\{(1,1,1,0),(0,0,1,1)\}$ independent part

$(a,b,2,0) = c_1(1,1,1,0) + c_2(0,0,1,1)$

$(a,b,2,0) = (c_1, c_1, c_1+c_2, c_2)$

$a = c_1$, $\boxed{a=2}$;
$b = c_1$, $\boxed{b=2}$
$c_1+c_2 = 2$, $\boxed{c_1=2}$
$c_2 = 0$

$(0,0,c,d) = (c_3, c_3, c_3+c_4, c_4)$
$0 = c_3$
$c = c_3 + c_4$
$\boxed{c_4 = d}$ $\boxed{d=3}$;
$\boxed{c_4 = c}$ $\boxed{c=3}$

$(h,4,k,0) = (c_5, c_5, c_5+c_6, c_6)$
$+ \boxed{h = c_5 = 4}$ $\boxed{h=4}$
$+ c_5 = 4$
$+ c_5 + c_6 = k$; $\boxed{k=4}$
$+ c_6 = 0$

Write the Col(F) as span of some independent columns.

Col(F) = span$\{(2,0,4), (2,3,4)\}$

$\rightarrow \begin{bmatrix} 2 & 0 & 4 \\ 2 & 3 & 4 \end{bmatrix} \xrightarrow{-R_1+R_2 \to R_2} \begin{bmatrix} 2 & 0 & 4 \\ 0 & 3 & 0 \end{bmatrix}$ independent

QUESTION 5. (4 points) Let $D = \{\begin{bmatrix} a+2b & 2a+4b \\ 0 & 3a+6b \end{bmatrix} \mid a, b \in R\}$. Convince me that D is a subspace of $R^{2\times 2}$, then find the independent number of D (i.e., $IN(D)$, another name $dim(D)$).

can be written as span

$R^{2\times2} \leftrightarrow \mathbb{R}^4$

$D = \left\{ a \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} + b \begin{bmatrix} 2 & 4 \\ 0 & 6 \end{bmatrix} \;\middle|\; a, b \in \mathbb{R} \right\}$

$+ \; D = \text{span}\left\{ \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 4 \\ 0 & 6 \end{bmatrix} \right\}$ + Since it can be written in the language of span, it is a subspace of 2×2

$R^{2\times2} \leftrightarrow \mathbb{R}^4$
+

$\begin{array}{c} f_1 \\ f_2 \end{array} \begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 4 & 0 & 6 \end{bmatrix} \xrightarrow{-2R_1+R_2 \to R_2} \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

zeroes row + f_2 depends on f_1
the system is dependent

+ $\dim(D) = 1$

+ it's more practical to write D as span$\left\{ \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \right\}$

QUESTION 6. (4 points) Let $D = \{f(x) \in P_4 | f'(1) = 0\}$. Convince me that D is a subspace of P_4, then find the independent number of D (i.e., $IN(D)$, another name $dim(D)$).

$P_4 \longleftrightarrow R_4$

a polynomial living in P_4

$a_3x^3 + a_2x^2 + a_1x + a_0$

$f'(1) = 3a_3(1)^2 + 2a_2(1) + a_1 + 0$

$+ \boxed{f'(1) = 3a_3 + 2a_2 + a_1 + 0 = 0}$; $a_1 = -3a_3 - 2a_2$

$D = \{a_3x^3 + a_2x^2 + a_1x + a_0 \mid \text{where } a_1 = -3a_3 - 2a_2,\ a_0 = 0 \text{ such that } a_3, a_2 \in \mathbb{R}\}$

$D = \{a_3x^3 + a_2x^2 + (-3a_3 - 2a_2)x + 0 \mid a_3, a_2 \in \mathbb{R}\}$ a_0

$D = \{a_3x^3 + a_2x^2 - 3a_3x - 2a_2x + 0 \mid a_3, a_2 \in \mathbb{R}\}$

λ

$D = \{a_3(x^3 - 3x) + a_2(x^2 - 2x) + a_0(1)\} \mid a_3, a_2 \in \mathbb{R}$

subspace

$+ D = \text{span}(x^3 - 3x, x^2 - 2x, 1)$

$$\begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & -2 & 0 \end{bmatrix}$$

$dim(D) = 3$

$-2R_1 + R_3 \to R_3$ $\quad R_3 \leftrightarrow R_1$

QUESTION 7. (6 points) Given A is a 3×4 matrix such that $A \xrightarrow{2R_1 + R_3 \to R_3} B \xrightarrow{R_3 \leftrightarrow R_1} C$.

a) Find two elementary matrices say E, F such that $EFA = C$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

b) Find two elementary matrices say F, W such that $FWC = A$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

c) Use row operations only to determine the matrix B, where $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix} = B$

$2R_1$ $\quad$ elementary matrices $\quad R_1 \leftrightarrow R_2$ $\quad -2R_2 + R_1 \to R_1$

$$\begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix} \xrightarrow{-2R_2 + R_1 \to R_1} \begin{bmatrix} -3 & -7 \\ 2 & 4 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 2 & 4 \\ -3 & -7 \end{bmatrix}$$

$$\xrightarrow{2R_1} = \begin{bmatrix} 4 & 8 \\ -3 & -7 \end{bmatrix} = B$$

QUESTION 8. (3 points) Let A be a matrix 2×3 such that $Rank(A) = 2$. Someone told you that I can construct such matrix such that $A \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $A \begin{bmatrix} -3 \\ -12 \\ 7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. You smiled and you said "if you live all your life + life after you will never be able to construct such matrix, because ... ". What is the reason that you gave? I'll give an example (math argument)

Consider the homogenous system AX = O (where O is a 2X 1 zero-column). Since Rank(A) = 2, the system will have two leading variable and one free variable. Thus the solution set of the homogeneous system, call it F, we have F = span{one non-zero point in R^3} and it has independent number 1 (i.e., it has dimension 1). Now from the hypothesis the points Q =(1,4, 2) and W = (-3, -12, 7) "live" in D. Since IN(F) = 1, Q and W must be dependent. However, if we check..they are INDEPENDENT (impossible). Hence such matrix does not exist